

# ON THE EXISTENCE OF COMPLEX EIGENVALUES NEAR THE ISOLATED ESSENTIAL SPECTRUM

DIOMBA SAMBOU

**ABSTRACT.** We prove complex eigenvalue upper bounds and asymptotics for non-self-adjoint relatively compact perturbations of certain operators of mathematical physics. In particular, these asymptotics describe the distribution (the rate convergence) and prove the existence of the complex eigenvalues in a neighborhood of the essential spectrum of the operators. For instance, we apply our results to quantum Landau Hamiltonians  $(-i\nabla - A)^2 - b$  with constant magnetic field of strength  $b > 0$ . We obtain the main asymptotic term of the complex eigenvalues counting function for an annulus centered at a Landau level  $2bq$ ,  $q \in \mathbb{N}$ . On this way, we prove that they are localized in certain sectors adjoining the Landau levels and that they accumulate to these thresholds asymptotically.

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## 1. INTRODUCTION AND ABSTRACT RESULTS

**1.1. Introduction.** First, let us introduce some conventional notations and definitions that will be used in this article. For a closed operator  $Z$  acting on a separable Hilbert space  $X$ , its resolvent will be denoted by  $\mathcal{R}(\lambda)$ . If  $\lambda$  is an isolated point of  $\sigma(Z)$ , the spectrum of  $Z$ , the Riesz projection of  $Z$  with respect to  $\lambda$  is defined by

$$P_\lambda := \frac{1}{2i\pi} \int_\gamma \mathcal{R}(z) dz.$$

Here  $\gamma$  is a small positively oriented circle centered at  $\lambda$ , containing  $\lambda$  as the only point of  $\sigma(Z)$ . We shall say that  $\lambda$  is a discrete eigenvalue of  $Z$  if its algebraic multiplicity

$$(1.1) \quad \text{mult}(\lambda) := \text{rank}(P_\lambda)$$

is finite. Noting that  $\text{mult}(\lambda) \geq \text{rank}(\ker(Z - \lambda))$ , the geometric multiplicity of  $\lambda$ , with equality if  $Z$  is self-adjoint. So, the discrete spectrum of  $Z$  denoted by  $\sigma_{\text{disc}}(Z)$  is the set of discrete eigenvalues of  $Z$ . We define the essential spectrum of  $Z$  as the set of complex numbers  $\lambda$  such that  $Z - \lambda$  is not a Fredholm operator. It will be denoted by  $\sigma_{\text{ess}}(Z)$  and it is a closed subset of  $\sigma(Z)$ . Recall that a compact linear operator  $L$  belongs to the Schatten-von Neuman class  $\mathcal{S}_p(X)$ ,  $p \in [1, +\infty)$ , if the norm  $\|L\|_{\mathcal{S}_p} := (\text{Tr } |L|^p)^{1/p}$  is finite. If  $p = \infty$ ,  $\mathcal{S}_\infty(X)$  is the set of compact linear operators on  $X$ .

Most of the known results on the discrete spectrum of magnetic quantum Hamiltonians deal with self-adjoint electric potentials  $W$ . In particular, if  $W$  admits power-like or slower decay at infinity, or if  $W$  is compactly supported, the behaviour of the discrete spectrum near boundary points of the essential spectrum has been extensively studied, [16, Chap. 11-12], [20], [21], [22], [19], [29], [30]. For a recent survey on this topic, see [24]. Still, there are relatively few results concerning non-self-adjoint electric potentials and most of them give Lieb-Thirring type inequalities, [11], [5], [4], [7], [8], [15], [12], [32], [26], [27], [10]. See [32] for a recent survey on this topic.

The purpose of this present article is to establish new results of upper bounds and asymptotic behaviours on the discrete spectrum of a general class of self-adjoint operators (see (1.2) and (1.3) below) perturbed by relatively compact non-self-adjoint electric potentials. In particular, we derive from the asymptotic behaviours the existence of infinite complex eigenvalues near boundary points of the essential spectrum of the operators. The main point is that the current construction applies to a large class of operators containing the Landau and Dirac Hamiltonians of full rank. In particular, we establish in some sens an extension to non-self-adjoint case of results by Raikov-Warzel [23] that treat self-adjoint electric potentials.

Let  $\mathcal{H}_0$  be an unbounded self-adjoint operator defined on a dense subset of  $X$ . Assume that its spectrum  $\sigma(\mathcal{H}_0)$  is given by an infinite discrete sequence of (real) eigenvalues of infinite multiplicity, i.e.

$$(1.2) \quad \sigma(\mathcal{H}_0) = \sigma_{\text{ess}}(\mathcal{H}_0) = \bigcup_{q=0}^{\infty} \{\Lambda_q\},$$

where  $\Lambda_0 \geq 0$  and  $\Lambda_{q+1} > \Lambda_q$ . The orthogonal projection onto  $\ker(\mathcal{H}_0 - \Lambda_q)$  will be denoted by  $p_q$ .

**Remark 1.1.** Noting that in more general considerations, our abstract results Theorems 1.2 and 1.3 remain valid if instead (1.2) we assume that the spectrum of the operator  $\mathcal{H}_0$  is a disjoint union of a finite number of isolated thresholds  $\Lambda_q$ ,  $q \geq 0$ , and an absolute continuous part  $[\zeta, +\infty)$ . Namely,

$$(1.3) \quad \sigma(\mathcal{H}_0) = \sigma_{\text{ess}}(\mathcal{H}_0) = \bigcup_{q=0}^N \{\Lambda_q\} \amalg [\zeta, +\infty),$$

where the spectral thresholds  $\Lambda_q$  are as in (1.2) with  $\zeta > \Lambda_N$  for some fixed integer  $N < \infty$ .

On the domain of the operator  $\mathcal{H}_0$ , we introduce the perturbed operator

$$(1.4) \quad \mathcal{H} = \mathcal{H}_0 + W,$$

where  $W$  is a bounded non-self-adjoint electric potential which does not vanish identically. For  $\lambda \in \rho(\mathcal{H}_0)$ , the resolvent set of  $\mathcal{H}_0$ , we also require that the weighted resolvent

$$(1.5) \quad |W|^{\frac{1}{2}}(\mathcal{H}_0 - \lambda)^{-1} \in \mathcal{S}_p,$$

for some  $p \geq 2$ . Which is a stronger condition implying that the electric potential  $W$  is relatively compact with respect to the operator  $\mathcal{H}_0$ . Then it follows from the Weyl's criterion concerning the invariance of the essential spectrum that  $\sigma_{\text{ess}}(\mathcal{H})$  coincides with  $\sigma_{\text{ess}}(\mathcal{H}_0) = \bigcup_{q=1}^{\infty} \{\Lambda_q\}$ . However, the electric potential  $W$  may generate (complex) discrete eigenvalues that can only accumulate to  $\sigma_{\text{ess}}(\mathcal{H}) = \bigcup_{q=0}^{\infty} \{\Lambda_q\}$ , see [14, Theorem 2.1, p. 373]. A problem that arise, is to precise the rate of this accumulation by studying the distribution of the discrete spectrum  $\sigma_{\text{disc}}(\mathcal{H})$  of  $\mathcal{H}$  near the spectral thresholds  $\Lambda_q$ . Motivated by this question, in a recent work by the author [26], [27], the following result often called a generalized Lieb-Thirring type inequality (see Lieb-Thirring [18] for original work), and formulated in the present context is obtained.

**Theorem 1.1.** [27, Theorem 1.1] *Let  $\mathcal{H}_0$  be a self-adjoint operator such that  $\sigma(\mathcal{H}_0) = \bigcup_{q=1}^{\infty} \{\Lambda_q\}$  as above with  $|\Lambda_{q+1} - \Lambda_q| \leq \delta$ , a fixed positive constant. Let  $\mathcal{H} := \mathcal{H}_0 + W$ , and for some  $p > 1$ , assume that the potential  $W$  satisfies*

$$(1.6) \quad \|W(\mathcal{H}_0 - \mu_0)^{-1}\|_{\mathcal{S}_p}^p \leq K_0,$$

with  $K_0$  a positive constant and  $\mu_0 := -\|W\|_{\infty} - 1$ . Then the following holds

$$(1.7) \quad \sum_{\lambda \in \sigma_{\text{disc}}(\mathcal{H})} \frac{\text{dist}(\lambda, \bigcup_{q=1}^{\infty} \{\Lambda_q\})^p}{(1 + |\lambda|)^{2p}} \leq C_0 K_0 (1 + \|V\|_{\infty})^{2p},$$

where  $C_0 = C(p, \Lambda_0)$  is a constant depending on  $p$  and  $\Lambda_0$ .

Now, let us discuss briefly about the above theorem. Assume that there exists  $(\lambda_{\ell})_{\ell} \subset \sigma_{\text{disc}}(\mathcal{H})$  a sequence of complex eigenvalues that converges to

a point of  $\sigma_{\text{ess}}(\mathcal{H}) = \cup_{q=0}^{\infty} \{\Lambda_q\}$ . Then (1.7) implies that

$$(1.8) \quad \sum_{\ell} \text{dist}(\lambda_{\ell}, \cup_{q=0}^{\infty} \{\Lambda_q\})^p < \infty.$$

This means that, *a priori*, the complex eigenvalues from  $\sigma_{\text{disc}}(\mathcal{H})$  are getting less densely distributed in a neighborhood of the  $\Lambda_q$  with growing  $p$ . However, even if Theorem 1.1 allows to derive formally the rate accumulation (or convergence) of the complex eigenvalues near the spectral thresholds  $\Lambda_q$ ,  $q \geq 0$ , it does not prove their existence.

In this present article, we answer positively to the problem of existence of complex eigenvalues of the operator  $\mathcal{H}$  by considering the class of bounded electric potentials  $W$  satisfying assumption (1.5). First, we establish a sharp upper bound on the number of the discrete eigenvalues around a fixed threshold  $\Lambda_q$ . Formally, this corresponds to the case  $p = 0$  in the generalized Lieb-Thirring type inequality (1.7) in Theorem 1.1. To get the upper bound, we use techniques close to those from [2] and [25]. Second, under appropriate assumptions and using characteristic values tools of a meromorphic function initiated and developed by Bony-Bruneau-Raikov [3], we obtain the main asymptotic term of the complex eigenvalues counting function for an annulus centered at a threshold  $\Lambda_q$ . In particular, we establish the existence of infinite complex eigenvalues in some sectors adjoining  $\Lambda_q$ . Roughly speaking, our methods can be considered as a Birman-Schwinger principle applied to the non-self-adjoint perturbed operator  $\mathcal{H}$ , see Proposition 3.2 below. On this way, for a fixed threshold  $\Lambda_q$ , we reduce the study of the distribution of the complex eigenvalues near  $\Lambda_q$  to the analyze of the Berezin-Toeplitz operator  $p_q|W|p_q$ .

**1.2. Formulation of abstract results.** In order to formulate ours abstract results, let us first introduce some notations. For a fixed threshold  $\Lambda_q$ ,  $q \in \mathbb{N} := \{0, 1, 2, \dots\}$ , and  $\varepsilon > 0$ , let  $D_q(\varepsilon)^*$  be the pointed disk defined by

$$(1.9) \quad D_q(\varepsilon)^* := \{\lambda \in \mathbb{C} : 0 < |\Lambda_q - \lambda| < \varepsilon\}.$$

Let us put the change of variables  $\Lambda_q - \lambda = k$  and introduce also the pointed disk  $D(0, \varepsilon)^*$  by

$$(1.10) \quad D(0, \varepsilon)^* := \{k \in \mathbb{C} : 0 < |k| < \varepsilon\}.$$

Then, the disk  $D_q(\varepsilon)^*$  defined by (1.9) can be parametrized by  $\lambda = \lambda_q(k) := \Lambda_q - k$  with  $k \in D(0, \varepsilon)^*$ . Note that we have

$$(1.11) \quad D_q(\varepsilon)^* = \Lambda_q + D(0, \varepsilon)^*$$

for any  $q \geq 0$ . In what follows below, the radius  $\varepsilon$  will be assumed sufficiently small. So, we go to the following upper bound on the number of discrete eigenvalues in a vicinity of a threshold  $\Lambda_q$ ,  $q \in \mathbb{N}$ .

**Theorem 1.2. Upper bound.** *Assume that (1.5) is satisfied for some  $p \geq 2$ . Then in small annulus, there exists  $r_0 > 0$  such that for any  $0 < r < r_0$ ,*

$$(1.12) \quad \sum_{\substack{\lambda_q(k) \in \sigma_{\text{disc}}(\mathcal{H}) \\ r < |k| < 2r}} \text{mult}(\lambda_q(k)) = \mathcal{O}\left(\text{Tr} \mathbf{1}_{(r, \infty)}(p_q|W|p_q) |\ln r|\right),$$

where the multiplicity  $\text{mult}(\lambda_q(k))$  of the discrete eigenvalue  $\lambda_q(k) := \Lambda_q - k$  is defined by (1.1).

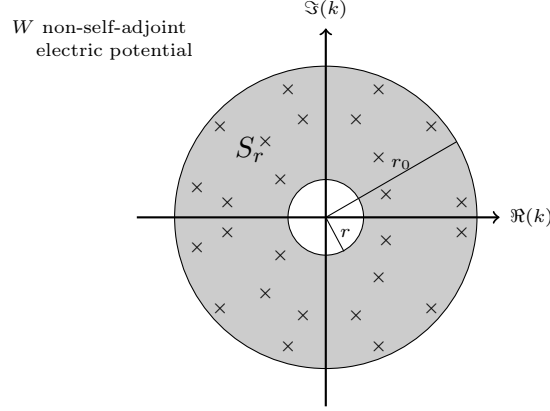


FIGURE 1.1. **Complex eigenvalues near a threshold  $\Lambda_q$ :**

For  $r_0 < \varepsilon$  sufficiently small, the number of complex eigenvalues  $\lambda_q(k) = \Lambda_q - k$  of  $\mathcal{H} := \mathcal{H}_0 + W$  in the annulus  $S_r$  is bounded by  $\mathcal{O}\left(\text{Tr} \mathbf{1}_{(r, \infty)}(p_q |W| p_q) |\ln r|\right)$  for general potentials  $W$ , see Theorem 1.2.

**Remark 1.2.** In some concrete examples with  $\mathcal{H}_0$  the Landau Hamiltonians, eigenvalue asymptotics for the Berezin-Toeplitz operator  $p_q |W| p_q$  can be described when  $|W|$  admits a power-like decay, exponential decay, or is compactly supported, see (2.8), (2.9) and (2.10) below.

To formulate our second main abstract result, we need to consider non-self-adjoint electric potentials  $W$  of the form

$$(1.13) \quad W = e^{i\alpha} V \quad \text{with} \quad \alpha \in \mathbb{R} \quad \text{and} \quad V : \text{Dom}(\mathcal{H}_0) \longrightarrow L^2(\mathbb{R}^2)$$

is the multiplication operator by the function  $V : \mathbb{R}^2 \longrightarrow \mathbb{R}$ . Let  $J := \text{sign}(V)$  be the sign of  $V$ . For  $k \in D(0, \varepsilon)^*$  and a given threshold  $\Lambda_q$ , let us introduce the operator

$$(1.14) \quad \mathcal{A}_q(k) := -J|V|^{\frac{1}{2}} \left( p_q + k \sum_{j \neq q} p_j (\mathcal{H}_0 - \Lambda_q + k)^{-1} \right) |V|^{\frac{1}{2}}.$$

Under the above considerations, clearly the operator  $\mathcal{A}_q(k)$  is holomorphic on the disk  $D(0, \varepsilon)^*$  if the radius  $\varepsilon$  is chosen such that

$$(1.15) \quad \varepsilon < \inf_{j \neq q} |\Lambda_q - \Lambda_j|.$$

Let  $\Pi_q$  be the orthogonal projection onto  $\ker \mathcal{A}_q(0)$ , and for  $V$  of definite sign  $J = \pm$ , introduce the following condition:

$$(1.16) \quad I - e^{i\alpha} \mathcal{A}'_q(0) \Pi_q \text{ is an invertible operator.}$$

**Remark 1.3.** Note that assumption (1.16) is generically satisfied. Namely, it remains valid for  $e^{i\alpha}$  in the complement of a discrete set formed by the inverses of the non-vanishing eigenvalues of the compact operator  $\mathcal{A}'_q(0) \Pi_q$ .

For  $r_0, \delta$ , two positive constants fixed, and  $r > 0$  tending to zero, let us define the sector  $\mathcal{C}_\delta(r, r_0)$  by

$$(1.17) \quad \mathcal{C}_\delta(r, r_0) := \{x + iy \in \mathbb{C} : r \leq x \leq r_0, -\delta x \leq y \leq \delta x\}.$$

**Theorem 1.3. Localization, asymptotic expansions.** *Let  $W$  satisfying (1.13) for some  $\alpha \in \mathbb{R}$ , and  $V$  of definite sign  $J = \pm$ . Assume that (1.16) happens. If (1.5) holds for some  $p \geq 2$ , then in small annulus there exists  $r_0 > 0$  such that the operator  $\mathcal{H} := \mathcal{H}_0 + W$  has the following properties.*

(i) *For any  $\delta > 0$  and for  $|k| < r_0$ , the discrete eigenvalues  $\lambda_q(k) = \Lambda_q - k$  of  $\mathcal{H}$  satisfy*

$$(1.18) \quad \lambda_q \in \Lambda_q \pm e^{i\alpha} \mathcal{C}_\delta(r, r_0),$$

where  $\mathcal{C}_\delta(r, r_0)$  is the sector defined by (1.17).

(ii) *Assume that  $\text{Tr} \mathbf{1}_{(r, \infty)}(p_q | V | p_q) \rightarrow +\infty$  as  $r \searrow 0$ . Then, there exists a sequence  $(r_\ell)_\ell$  tending to zero such that*

$$(1.19) \quad \sum_{\substack{\lambda_q(k) \in \sigma_{\text{disc}}(\mathcal{H}) \\ r_\ell < |k| < r_0}} \text{mult}(\lambda_q(k)) = \text{Tr} \mathbf{1}_{(r_\ell, \infty)}(p_q | V | p_q) (1 + o(1))$$

as  $\ell \rightarrow \infty$ .

(iii) *If we have  $\text{Tr} \mathbf{1}_{(r, \infty)}(p_q | V | p_q) = \phi(r)(1 + o(1))$  as  $r \searrow 0$ , where the function  $\phi(r)$  is such that*

$$\phi(r(1 \pm \nu)) = \phi(r)(1 + o(1) + \mathcal{O}(\nu))$$

for any  $\nu > 0$  small enough, then

$$(1.20) \quad \sum_{\substack{\lambda_q(k) \in \sigma_{\text{disc}}(\mathcal{H}) \\ r < |k| < r_0}} \text{mult}(\lambda_q(k)) = \text{Tr} \mathbf{1}_{(r, \infty)}(p_q | V | p_q) (1 + o(1))$$

as  $r \searrow 0$ .

**Remark 1.4.** Points (ii) and (iii) of Theorem 1.3 prove the existence of complex eigenvalues in the sector  $S_\alpha^\theta$  (see FIGURE 1.2) and their accumulation to  $\Lambda_q$ . Point (i) of Theorem 1.3 says just that for  $\pm V > 0$ , the complex eigenvalues  $\lambda_q(k) = \Lambda_q - k$  of the operator  $\mathcal{H}$  near a threshold  $\Lambda_q$  are concentrated around the semi-axis  $\lambda_q \in \Lambda_q \pm e^{i\alpha}]0, +\infty)$ . In particular, the case  $e^{i\alpha} = 1$  coincides with self-adjoint electric potentials  $W$  of definite sign, and condition (1.18) becomes  $\lambda_q \in \Lambda_q \pm ]0, +\infty)$ . This means that the discrete eigenvalues  $\lambda_q(k)$  are localized respectively on the right and the left of the threshold  $\Lambda_q$ . However, in the case  $e^{i\alpha} = i$ , condition (1.18) can be rewritten, see (3.18), as  $\mp \Im(k) \geq 0$  with  $|\Re(k)| = o(|k|)$ . Which is equivalent to the fact that the complex eigenvalues  $\lambda_q(k)$  are concentrated around the semi-axis  $\lambda_q \in \Lambda_q \pm i]0, +\infty)$ .

The paper is organised as follows. In Section 2, we apply our abstract results to (magnetic) Landau Hamiltonians. On this way, we prove that our abstract assumptions above are satisfied in certain concrete examples. Section 3 concerns the strategy of the proof of our results. In Subsection 3.1, we give a preliminary analyse of the discrete eigenvalues in a neighborhood

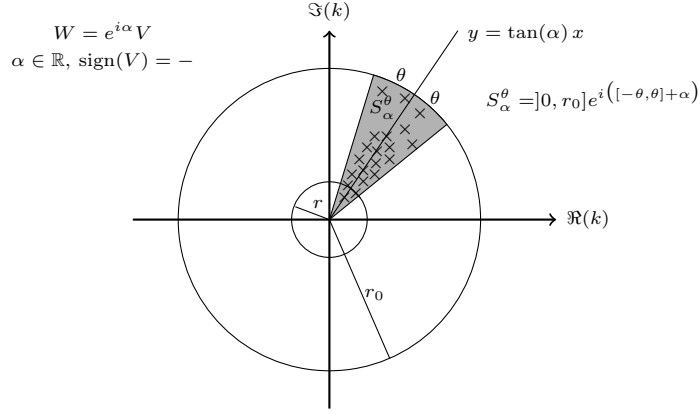


FIGURE 1.2. **Localization of the complex eigenvalues near a spectral threshold  $\Lambda_q$  with respect to the variable  $k$ :** For  $r_0$  sufficiently small, the discrete eigenvalues  $\lambda_q(k) = \Lambda_q - k$  of  $\mathcal{H} := \mathcal{H}_0 + e^{i\alpha}V$  are localized around the semi-axis  $k \in e^{i\alpha}]0, +\infty)$ , see Theorem 1.3. In particular, for  $e^{i\alpha} = \pm 1$ , the eigenvalues are real and are localized in the semi-axis  $k \in \pm]0, +\infty)$ .

of a fixed spectral threshold  $\Lambda_q$ ,  $q \in \mathbb{N}$ . Subsection 3.2 and Subsection 3.3 are devoted to the proofs of our abstract results. For the transparency of the presentation, we present in the Appendix auxiliary tools we need as characteristic values of a meromorphic function [3], and the index of a finite meromorphic function.

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## 2. APPLICATION TO LANDAU HAMILTONIANS

In this Section, we apply our abstract results to (magnetic) Landau Hamiltonians. Concretely, our results can be formulated in more general considerations, see Remark 2.1 below. However, to simplify we shall consider only the two-dimensional case.

**2.1. Preliminaries.** Set  $x := (x_1, x_2) \in \mathbb{R}^2$ , and introduce the self-adjoint Landau Hamiltonian  $H_0$  on  $L^2(\mathbb{R}^2) := L^2(\mathbb{R}^2, \mathbb{C})$  by

$$(2.1) \quad H_0 := \left( D_{x_1} + \frac{bx_2}{2} \right)^2 + \left( D_{x_2} - \frac{bx_1}{2} \right)^2 - b,$$

where the constant  $b > 0$  is the strength of the magnetic field, and  $D_\nu := -i\frac{\partial}{\partial \nu}$ . The operator  $H_0$  is originally defined on  $C_0^\infty(\mathbb{R}^2)$ , and then closed

in  $L^2(\mathbb{R}^2)$ . It is well known, see e.g. [9], that the spectrum  $\sigma(H_0)$  of  $H_0$  is given by the Landau levels

$$(2.2) \quad \Lambda_q = 2bq, \quad q \in \mathbb{N},$$

and each  $\Lambda_q$  is an eigenvalue of infinite multiplicity.

On the domain of  $H_0$ , define the perturbed operator

$$(2.3) \quad H = H_0 + W,$$

where  $W$  is a non-self-adjoint electric potential identified with the multiplication operator by the function  $W : \mathbb{R}^2 \rightarrow \mathbb{C}$ . In the sequel, we adopt the notations and terminologies from Section 1. First, let us consider electric potentials  $W$  satisfying

$$(2.4) \quad 0 \not\equiv W, \quad W \in L^\infty(\mathbb{R}^2), \quad W \in L^{\frac{p}{2}}(\mathbb{R}^2), \quad p \geq 2.$$

The following lemma applied to the potential  $W$  satisfying (2.4) shows that (1.5) happens with  $\mathcal{H}_0 = H_0$ .

**Lemma 2.1.** [26, Lemma 6.1] *Let  $G \in L^p(\mathbb{R}^2)$  for some  $p \geq 2$ . Then for any  $\lambda \in \mathbb{C} \setminus \cup_{q=0}^\infty \{\Lambda_q\}$ , the operator  $G(H_0 - \lambda)^{-1} \in \mathcal{S}_p$  with*

$$(2.5) \quad \|G(H_0 - \lambda)^{-1}\|_{\mathcal{S}_p}^p \leq \frac{C(1 + |\lambda|)\|G\|_{L^p}^p}{\text{dist}(\lambda, \cup_{q=0}^\infty \{\Lambda_q\})^p},$$

where  $C = C(p, b)$  is constant depending on  $p$  and  $b$ .

**2.2. Statement of the results.** Since  $W$  is bounded, then Lemma 2.1 above implies in particular that it is relatively compact with respect to the operator  $H_0$ . The following theorem is an immediate application of Theorem 1.2 with  $X = L^2(\mathbb{R}^2)$ ,  $\mathcal{H}_0 = H_0$ , and  $\mathcal{H} = H$ .

**Theorem 2.1. Upper bound.** *Assume that  $W$  satisfies (2.4) for some  $p \geq 2$ . Then there exists  $0 < r_0 < 2b$  such that for any  $0 < r < r_0$ ,*

$$(2.6) \quad \sum_{\substack{\lambda_q(k) \in \sigma_{\text{disc}}(H) \\ r < |k| < 2r}} \text{mult}(\lambda_q(k)) = \mathcal{O}\left(\text{Tr} \mathbf{1}_{(r, \infty)}(p_q | W | p_q) |\ln r|\right),$$

where the multiplicity  $\text{mult}(\lambda_q(k))$  of the discrete eigenvalue  $\lambda_q(k) := \Lambda_q - k$  is defined by (1.1).

**Remark 2.1.** In more general considerations, i.e. in higher even dimension  $n = 2d$ ,  $d \geq 1$ , we can consider the self-adjoint Landau Hamiltonian  $H_{\text{Landau}}$  in  $L^2(\mathbb{R}^n)$  with constant magnetic field of full rank. It has the form  $(-i\nabla - A)^2$ , where  $A := (A_1, \dots, A_n)$  is the magnetic potential generating the magnetic field. In appropriate coordinates  $(x_1, y_1, \dots, x_d, y_d) \in \mathbb{R}^n$ , the operator  $H_{\text{Landau}}$  can be written as

$$H_{\text{Landau}} = \sum_{j=1}^d \left\{ \left( -i \frac{\partial}{\partial x_j} + \frac{b_j y_j}{2} \right)^2 + \left( -i \frac{\partial}{\partial y_j} - \frac{b_j x_j}{2} \right)^2 \right\},$$

where  $b_1 \geq \dots \geq b_d > 0$  are real numbers. The operator  $H_0$  considered in this article and defined by (2.1) is just a shifted Landau Hamiltonian in the special case  $d = 1$ ,  $b_1 = b_2 = b$ . For the physical applications of the



operator  $H_{\text{Landau}}$ , situation  $n = 2$  is the most frequently used case. However, Theorem 2.1 remains valid in higher dimensions. Note that the spectrum of the operator  $H_{\text{Landau}}$  is given by a discrete set of increasing Landau levels, see e.g. [9]. In particular, if  $b_1 = \cdots b_d = b$ , then the Landau levels are given by  $\Lambda_q = b(2q + 1)$ ,  $q \in \mathbb{N}$ .

Typically, assumption (2.4) above is satisfied by the special class of electric potentials  $W : \mathbb{R}^2 \rightarrow \mathbb{C}$  satisfying the estimate

$$(2.7) \quad |W(x)| \leq C \langle x \rangle^{-m}, \quad m > 0, \quad pm > 4, \quad p \geq 2,$$

for some positive constant  $C$  and  $\langle x \rangle := (1 + |x|^2)^{1/2}$ . If the function  $|W| = U$  admits a power-like decay, exponential decay, or is compactly supported, then asymptotic expansions as  $r \searrow 0$  of the quantity  $\text{Tr} \mathbf{1}_{(r, \infty)}(p_q U p_q)$  are well known. We quote to the following three types of assumptions on  $U$ :

**(A1)**  $U \in C^1(\mathbb{R}^2)$  satisfying the asymptotic property

$$U(x) = u_0(x/|x|)|x|^{-m}(1 + o(1)), \quad |x| \rightarrow \infty,$$

where  $u_0$  is a non negative continuous function on  $\mathbb{S}^1$  which does not vanish identically, and

$$|\nabla U(x)| \leq C_1 \langle x \rangle^{-m-1}, \quad x \in \mathbb{R}^2,$$

with some constants  $m > 0$  and  $C_1 > 0$ . Then by [21, Theorem 2.6],

$$(2.8) \quad \text{Tr} \mathbf{1}_{(r, \infty)}(p_q U p_q) = C_m r^{-2/m}(1 + o(1)), \quad r \searrow 0,$$

where

$$C_m := \frac{b}{4\pi} \int_{\mathbb{S}^1} u_0(t)^{2/m} dt.$$

Let us mention that in [21, Theorem 2.6], (2.8) is given in a general version containing higher even dimensions  $n = 2d$ ,  $d \geq 1$ .

**(A2)**  $U \in L^\infty(\mathbb{R}^2)$  and satisfies

$$\ln U(x) = -\mu |x|^{2\beta}(1 + o(1)), \quad |x| \rightarrow \infty,$$

with some constants  $\beta > 0$  et  $\mu > 0$ . Then by [23, Lemma 3.4],

$$(2.9) \quad \text{Tr} \mathbf{1}_{(r, \infty)}(p_q U p_q) = \varphi_\beta(r)(1 + o(1)), \quad r \searrow 0,$$

where for  $0 < r < e^{-1}$ , we set

$$\varphi_\beta(r) := \begin{cases} \frac{1}{2} b \mu^{-1/\beta} |\ln r|^{1/\beta} & \text{si } 0 < \beta < 1, \\ \frac{1}{\ln(1+2\mu/b)} |\ln r| & \text{si } \beta = 1, \\ \frac{\beta}{\beta-1} (\ln |\ln r|)^{-1} |\ln r| & \text{si } \beta > 1. \end{cases}$$

**(A3)**  $U \in L^\infty(\mathbb{R}^2)$  has a compact support, and there exists a positive constant  $C$  satisfying  $C \leq U$  on an open non-empty subset of  $\mathbb{R}^2$ . Then by [23, Lemma 3.5],

$$(2.10) \quad \text{Tr} \mathbf{1}_{(r, \infty)}(p_q U p_q) = \varphi_\infty(r)(1 + o(1)), \quad r \searrow 0,$$

where

$$\varphi_\infty(r) := (\ln |\ln r|)^{-1} |\ln r|, \quad 0 < r < e^{-1}.$$

Extensions of [23, Lemma 3.4 and Lemma 3.5] in higher even dimensions are established in [19].

The following theorem is an immediate application of Theorem 1.3 with  $X = L^2(\mathbb{R}^2)$ ,  $\mathcal{H}_0 = H_0$ , and  $\mathcal{H} = H$ .

**Theorem 2.2. Localization, asymptotic expansions.** *Let  $W = e^{i\alpha}V$  satisfying (2.4),  $\alpha \in \mathbb{R}$  and  $V$  of definite sign  $\pm$ . If (1.16) holds, then in small annulus around each Landau level  $2bq$ ,  $H := H_0 + W$  has the following properties.*

(i) *There exist complex eigenvalues that accumulate near  $2bq$  and localized in sectors of the form  $S_\alpha^\theta = \Lambda_q \pm [0, r_0]e^{i[\alpha-\theta, \alpha+\theta]}$  for some  $0 < r_0 < 2b$ .*

(ii) *Eventually, if  $V$  satisfies assumptions (A1), (A2) or (A3), then*

$$(2.11) \quad \sum_{\substack{\lambda_q(k) \in \sigma_{\text{disc}}(H) \\ r < |k| < r_0}} \text{mult}(\lambda_q(k)) = \text{Tr} \mathbf{1}_{(r, \infty)}(p_q |V| p_q) (1 + o(1))$$

as  $r \searrow 0$ .

**Proof.** Point (i) is direct consequence of (i) and (ii) of Theorem 1.3.

Let us clarify point (ii). If we have  $\Phi(r) = r^{-\gamma}$ , or  $\Phi(r) = |\ln r|^\gamma$ , or  $\Phi(r) = (\ln |\ln r|)^{-1} |\ln r|$ , for some  $\gamma > 0$ , then it can be checked that

$$\phi(r(1 \pm \nu)) = \phi(r)(1 + o(1) + \mathcal{O}(\nu))$$

for any  $\nu > 0$  small enough. Hence it follows from (iii) of Theorem 1.3. ■

**Remark 2.2.** It goes without saying that Remark 1.4 and Remark 2.1 remain valid in the present situation. On the other hand, note that Theorem 2.1 is an extension to non-self-adjoint case of results by Raikov-Warzel [23] that treat self-adjoint potentials corresponding to the case  $e^{i\alpha} = \pm 1$ .

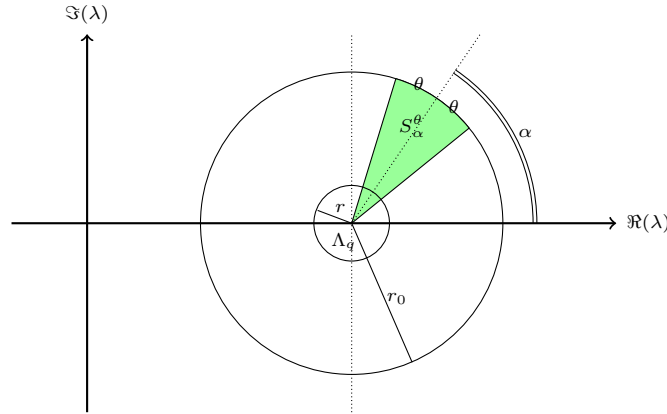


FIGURE 2.1. Sector  $S_\alpha^\theta = \Lambda_q \pm [0, r_0]e^{i[\alpha-\theta, \alpha+\theta]}$  adjoining a Landau level  $\Lambda_q = 2bq$  for  $r_0 < 2b$  small enough.

As an immediate consequence of Theorem 2.2, we have the following result.

**Theorem 2.3. Creation of complex eigenvalues.** *Let  $2bq$ ,  $q \in \mathbb{N}$  be a fixed Landau level. For any sector  $S_\alpha^\theta$  adjoining  $2bq$  as in Figure 2.1, there exists a non-self-adjoint electric potential  $W(\alpha, \theta)$  such that near  $2bq$ , all the discrete eigenvalues of  $H(\alpha, \theta) := H_0 + W(\alpha, \theta)$  are localized in  $S_\alpha^\theta$ . Moreover they accumulate to  $2bq$ .*

**2.3. Remark on Lieb-Thirring type inequalities.** Now, let us derive from Theorem 2.1 local Lieb-Thirring type inequalities around a fixed Landau level  $\Lambda_q$ ,  $q \in \mathbb{N}$ . The aim is to discuss the sharpness of the generalized Lieb-Thirring inequalities established in Sambou [26] for the Landau Hamiltonians.

Let  $(\lambda_\ell)_\ell$  be a sequence of complex eigenvalues converging to a Landau level  $\Lambda_q$ ,  $q \in \mathbb{N}$ . By point (i) of Theorem 2.2 induced by points (i) and (ii) of Theorem 1.3, each term  $\lambda_\ell$  of the sequence can be parametrized by  $\lambda_\ell = \Lambda_q - k_\ell$  with  $k_\ell \in \mp e^{i\alpha} \mathcal{C}_\delta(r, r_0)$ ,  $r \searrow 0$ , where the sector  $\mathcal{C}_\delta(r, r_0)$  is defined by (1.17). Then, each term  $k_\ell$  of the sequence  $(k_\ell)_\ell$  is such that  $k_\ell = \mp e^{i\alpha} \tilde{k}_\ell = \mp e^{i\alpha} (\mu_\ell + i\gamma_\ell)$  with  $|\gamma_\ell| \leq \delta |\mu_\ell|$ , where  $\mu_\ell$  and  $\gamma_\ell$  are respectively the real and the imaginary parts of  $\tilde{k}_\ell$ . Since for  $p \geq 2$  we have

$$(2.12) \quad \sum_\ell \text{dist}(\lambda_\ell, \Lambda_q)^p = \sum_\ell |k_\ell|^p = \sum_\ell |\tilde{k}_\ell|^p,$$

then the following upper and lower bounds hold

$$(2.13) \quad \sum_\ell |\mu_\ell|^p \leq \sum_\ell \text{dist}(\lambda_\ell, \Lambda_q)^p \leq (1 + \delta^2)^{\frac{p}{2}} \sum_\ell |\mu_\ell|^p.$$

Without any loss of generality, we can consider a subsequence  $(\tilde{k}_\ell)_\ell$  and assume that  $\dots > |\tilde{k}_\ell| > |\tilde{k}_{\ell+1}| > \dots$ . So let us introduce  $n(r, r_0)$ , the counting function of the  $\mu_\ell$  lying in the interval  $(r, r_0)$ . Namely,

$$n(r, r_0) := \#\{\mu_\ell : r < \mu_\ell < r_0\}.$$

By a geometric argument, it can be easily checked that

$$(2.14) \quad n(r, r_0) = \#\{\tilde{k}_\ell : r(1 + \delta^2)^{1/2} < |\tilde{k}_\ell| < r_0(1 + \delta^2)^{1/2}\}.$$

This implies that the asymptotic behaviour of  $n(r, r_0)$  as  $r \searrow 0$  is similar to that of the counting function of the  $k_\ell$  near zero. More precisely, equality (2.14) together with Theorem 2.1 imply that  $n(r, r_0)$  is of order  $\text{Tr} \mathbf{1}_{(r(1+\delta^2)^{1/2}, \infty)}(p_q | V | p_q)$  as  $r \searrow 0$ . In particular, if the potential  $V$  satisfies assumption (A1), then we have

$$(2.15) \quad \sum_\ell |\mu_\ell|^p = \int_0^{r_0} p r^{p-1} n(r, r_0) dr < \infty$$

if  $pm > 2$ . So, from (2.13), (2.15) and assumption (2.4) on  $W$ , we deduce that the finiteness of the sum in (2.12) holds if  $p > \max(4/m, 2)$ . In Sambou [26], the same condition on  $p$  is obtained for the generalized Lieb-Thirring inequalities under the same assumption on  $V$  in the context of the present article. However, it is convenient to mention that in [26], the hypothesis  $W \in L^p(\mathbb{R}^2)$ ,  $p > \max(2/m, 2)$  suffices to get the finiteness of the sum in (2.12). Novelty here is, in particular, that Theorem 2.1 allows us to

give concretely the existence and the rate accumulation (or convergence) of the complex eigenvalues near the Landau levels. Such conclusions are formally implied by the generalized Lieb-Thirring inequalities established in [26]. Noting that estimates above imply that

$$(2.16) \quad \sum_{\ell} \text{dist}(\lambda_{\ell}, \Lambda_q)^p \asymp \int_0^{r_0} pr^{p-1} n(r, r_0) dr.$$

### 3. STRATEGY OF PROOFS

**3.1. Complex eigenvalues near the spectral thresholds  $\Lambda_q$ ,  $q \in \mathbb{N}$ .** In this subsection, we give a local characterization of the discrete eigenvalues  $\lambda \in D_q(\varepsilon)^*$ , the neighborhood of the threshold  $\Lambda_q$  defined by (1.9) for  $\varepsilon$  small enough. Notations and assumptions are those from Section 1.

Let  $D(0, \varepsilon)^*$  be the pointed disk defined by (1.10) with  $\varepsilon$  small enough satisfying (1.15). Recall that the discrete eigenvalues  $\lambda \in D_q(\varepsilon)^*$  can be parametrized by  $\lambda = \lambda_q(k) := \Lambda_q - k$  with  $k \in D(0, \varepsilon)^*$ .

Let  $\mathcal{H}_0$  be the unbounded operator defined by (1.2), and  $W$  be the non-self-adjoint bounded electric potential defined by (1.4). With respect to the polar decomposition of  $W$ , let us write  $W = J|W|$ . Hence, for any  $\lambda \in \mathbb{C} \setminus \cup_{q=1}^{\infty} \{\Lambda_q\}$ , we have

$$(3.1) \quad \begin{aligned} & J|W|^{\frac{1}{2}} (\mathcal{H}_0 - \lambda)^{-1} |W|^{\frac{1}{2}} \\ &= J|W|^{\frac{1}{2}} \left( p_q(\Lambda_q - \lambda)^{-1} + \sum_{j \neq q} p_j(\mathcal{H}_0 - \lambda)^{-1} \right) |W|^{\frac{1}{2}}. \end{aligned}$$

Then, for  $k \in D(0, \varepsilon)^*$ , (3.1) becomes

$$(3.2) \quad \begin{aligned} & J|W|^{\frac{1}{2}} (\mathcal{H}_0 - \lambda_q(k))^{-1} |W|^{\frac{1}{2}} \\ &= J|W|^{\frac{1}{2}} \left( p_q k^{-1} + \sum_{j \neq q} p_j(\mathcal{H}_0 - \Lambda_q + k)^{-1} \right) |W|^{\frac{1}{2}} \end{aligned}$$

So, the following proposition follows immediately.

**Proposition 3.1.** *Let  $D(0, \varepsilon)^*$  be the pointed disk defined by (1.10) with  $\varepsilon$  small enough. Assume that (1.5) happens for some  $p \geq 2$ . Then, the operator-valued function*

$$D(0, \varepsilon)^* \ni k \longmapsto \mathcal{T}_W(\lambda_q(k)) := J|W|^{\frac{1}{2}} (\mathcal{H}_0 - \lambda_q(k))^{-1} |W|^{\frac{1}{2}}$$

*is analytic with values in  $\mathcal{S}_p(X)$ .*

Now, since the potential  $W$  is bounded, then assumption (1.5) implies that the operator  $W(\mathcal{H}_0 - \lambda)^{-1}$  belongs to the class  $\mathcal{S}_p$ ,  $p \geq 2$ . So, let us

introduce the regularized determinant  $\det_{[p]}(I + W(\mathcal{H}_0 - \lambda)^{-1})$  defined by

$$(3.3) \quad \det_{[p]}(I + W(\mathcal{H}_0 - \lambda)^{-1}) := \prod_{\mu \in \sigma(W(\mathcal{H}_0 - \lambda)^{-1})} \left[ (1 + \mu) \exp \left( \sum_{k=1}^{[p]-1} \frac{(-\mu)^k}{k} \right) \right],$$

where  $[p] := \min \{n \in \mathbb{N} : n \geq p\}$ . Let  $\mathcal{H}$  be the perturbed operator defined by (1.4). It is well known, see e.g. [28, Chap. 9], that we have the characterization

$$(3.4) \quad \lambda \in \sigma_{\text{disc}}(\mathcal{H}) \Leftrightarrow f_p(\lambda) := \det_{[p]}(I + W(\mathcal{H}_0 - \lambda)^{-1}) = 0.$$

Moreover, if the operator  $W(\mathcal{H}_0 - \lambda)^{-1}$  is holomorphic on a domain  $\Omega$ , then so is the function  $f_p(\lambda)$  in  $\Omega$ , and the algebraic multiplicity of  $\lambda \in \sigma_{\text{disc}}(\mathcal{H})$  is equal to the order of  $\lambda$  as zero of the function  $f_p(\lambda)$ .

We have the following characterization of the discrete eigenvalues in a neighborhood of a fixed threshold  $\Lambda_q$ ,  $q \in \mathbb{N}$ . The index of a finite meromorphic operator valued function appearing in (3.5) is recalled in the Appendix, Subsection 4.1.

**Proposition 3.2.** *Let  $\mathcal{T}_W(\lambda_q(k))$  be the operator defined in Proposition 3.1. Then, for  $k_0 \in D(0, \epsilon)^*$  with  $\epsilon$  small enough, the following assertions are equivalent:*

- (i)  $\lambda_q(k_0) := \Lambda_q - k_0$  is a discrete eigenvalue of  $\mathcal{H}$ ,
- (ii)  $\det_{[p]}(I + \mathcal{T}_W(\lambda_q(k_0))) = 0$ ,
- (iii)  $-1$  is an eigenvalue of  $\mathcal{T}_W(\lambda_q(k_0))$ .

Moreover, the following equality happens

$$(3.5) \quad \text{mult}(\lambda_q(k_0)) = \text{Ind}_\gamma(I + \mathcal{T}_V(\lambda_q(\cdot))),$$

where  $\gamma$  is a small contour positively oriented containing  $k_0$  as the unique point  $k$  satisfying  $\lambda_q(k)$  is a discrete eigenvalue of  $\mathcal{H}$ .

**Proof.** (i)  $\Leftrightarrow$  (ii) follows immediately from (3.4) and the equality

$$\det_{[p]}(I + W(\mathcal{H}_0 - \lambda)^{-1}) = \det_{[p]}(I + J|W|^{\frac{1}{2}}(\mathcal{H}_0 - \lambda)^{-1}|W|^{\frac{1}{2}}).$$

The equivalence (ii)  $\Leftrightarrow$  (iii) is a direct consequence of the definition of  $\det_{[p]}(I + J|W|^{\frac{1}{2}}(\mathcal{H}_0 - \lambda)^{-1}|W|^{\frac{1}{2}})$ .

Let us prove (3.5). Let  $f_p(\lambda)$  be the function defined by (3.4). By the discussion just after (3.4), if  $\gamma'$  is a small contour positively oriented containing  $\lambda_q(k_0)$  as the unique discrete eigenvalue of  $\mathcal{H}$ , then we have

$$(3.6) \quad \text{mult}(\lambda_q(k_0)) = \text{ind}_{\gamma'} f_p.$$

Here, the right side hand of (3.6) is the index, defined by (4.1), of the holomorphic function  $f_p$  with respect to the contour  $\gamma'$ . Now (3.5) follows immediately from the equality

$$\text{ind}_{\gamma'} f_p = \text{Ind}_\gamma(I + \mathcal{T}_V(\lambda_q(\cdot))),$$

see e.g. the identity (2.6) of [3] for more details.  $\blacksquare$

**3.2. Proof of Theorem 1.2.** To prove Theorem 1.2, we first need to split in a neighborhood of a fixed threshold  $\Lambda_q$ ,  $q \in \mathbb{N}$ , the weighted resolvent  $\mathcal{T}_W(\lambda) := J|W|^{\frac{1}{2}}(\mathcal{H}_0 - \lambda)^{-1}|W|^{\frac{1}{2}}$  into two parts.

Let  $k \in D(0, \varepsilon)^*$ , the pointed disk defined by (1.10) with  $\varepsilon$  small enough. By (3.2), for  $\lambda_q(k) := \Lambda_q - k$  a discrete eigenvalue in a vicinity of  $\Lambda_q$ , we have

$$(3.7) \quad \mathcal{T}_W(\lambda_q(k)) := J \frac{|W|^{\frac{1}{2}} p_q |W|^{\frac{1}{2}}}{k} + J \sum_{j \neq q} |W|^{\frac{1}{2}} p_j (\mathcal{H}_0 - \Lambda_q + k)^{-1} |W|^{\frac{1}{2}}.$$

Hence, the following proposition holds.

**Proposition 3.3.** *Let  $D(0, \varepsilon)^*$  be the pointed disk defined by (1.10) with  $\varepsilon$  small enough. Assume that (1.5) happens for some  $p \geq 2$ . Then, for  $k \in D(0, \varepsilon)^*$ , we have*

$$(3.8) \quad \mathcal{T}_W(\lambda_q(k)) := J \frac{|W|^{\frac{1}{2}} p_q |W|^{\frac{1}{2}}}{k} + \mathcal{A}_q(k),$$

with respect to the polar decomposition  $W = J|W|$  and

$$\mathcal{A}_q(k) := J \sum_{j \neq q} |W|^{\frac{1}{2}} p_j (\mathcal{H}_0 - \Lambda_q + k)^{-1} |W|^{\frac{1}{2}} \in \mathcal{S}_\infty$$

is a holomorphic operator on  $D(0, \varepsilon)^*$ .

Noting that  $|W|^{\frac{1}{2}} p_q |W|^{\frac{1}{2}}$  is a self-adjoint positif compact operator. Moreover, for  $r > 0$ , we have,

$$(3.9) \quad \text{Tr} \mathbf{1}_{(r, \infty)} \left( |W|^{\frac{1}{2}} p_q |W|^{\frac{1}{2}} \right) = \text{Tr} \mathbf{1}_{(r, \infty)} (p_q |W| p_q).$$

Introduce

$$N(\mathcal{H}) := \{(\mathcal{H}f, f) : f \in \text{dom}(\mathcal{H}), \|f\|_{L^2} = 1\}$$

the numerical range of the operator  $\mathcal{H}$ . It is well known (see e.g. [6, Lemma 9.3.14]) that the spectrum  $\sigma(\mathcal{H})$  of  $\mathcal{H}$  satisfies

$$\sigma(\mathcal{H}) \subseteq \overline{N(\mathcal{H})}.$$

The second crucial tool of the proof of the theorem is the following proposition.

**Proposition 3.4.** *Let  $\Lambda_q$ ,  $q \in \mathbb{N}$ , be a fixed spectral threshold, and  $s_0 < \varepsilon$  be small enough. Then, for  $0 < s < |k| < s_0$ , we have,*

(i)  $\lambda_q(k) := \Lambda_q - k$  is a discrete eigenvalue of  $\mathcal{H}$  near  $\Lambda_q$  if and only if  $k$  is a zero of the determinant

$$(3.10) \quad \mathcal{D}(k, s) := \det(I + \mathcal{K}(k, s)),$$

where  $\mathcal{K}(k, s)$  is a finite-rank operator analytic with respect to  $k$ , and satisfying

$$\text{rank } \mathcal{K}(k, s) = \mathcal{O}\left(\text{Tr} \mathbf{1}_{(s, \infty)}(p_q |W| p_q) + 1\right), \quad \|\mathcal{K}(k, s)\| = \mathcal{O}(s^{-1}),$$

uniformly with respect to  $s < |k| < s_0$ .

(ii) Moreover, if  $\lambda_q(k_0) := \Lambda_q - k_0$  is a discrete eigenvalue of  $\mathcal{H}$ , then we have

$$(3.11) \quad \text{mult}(\lambda_q(k_0)) = \text{Ind}_\gamma(I + \mathcal{K}(\cdot, s)) = \text{m}(k_0),$$

where  $\gamma$  is chosen as in (3.5), and  $\text{m}(k_0)$  is the multiplicity of  $k_0$  as zero of  $\mathcal{D}(k, s)$ .

(iii) If  $\lambda_q(k)$  is not a discrete eigenvalue of  $\mathcal{H}$  with  $\text{dist}(\lambda_q(k), \overline{N(\mathcal{H})}) > \varsigma > 0$ , then the operator  $I + \mathcal{K}(k, s)$  is invertible and satisfies

$$\left\| (I + \mathcal{K}(k, s))^{-1} \right\| = \mathcal{O}(\varsigma^{-1}),$$

uniformly with respect to  $s < |k| < s_0$ .

**Proof.** (i) – (ii) Let us set  $\mathcal{B}_q := |W|^{\frac{1}{2}} p_q |W|^{\frac{1}{2}}$  which is a compact operator. Then, by Proposition 3.3, for  $s < |k| \leq s_0 < \epsilon$ , we have

$$\mathcal{T}_W(\lambda_q(k)) = \frac{J}{k} \mathcal{B}_q + \mathcal{A}_q(k).$$

Now, the operator-valued function  $k \mapsto \mathcal{A}_q(k)$  is analytic near zero with values in  $\mathcal{S}_\infty$ . Then, for  $s_0$  small enough, there exists a finite-rank operator  $\mathcal{A}_0$  independent of  $k$ , and  $\tilde{\mathcal{A}}(k) \in \mathcal{S}_\infty$  analytic near zero with  $\|\tilde{\mathcal{A}}(k)\| < \frac{1}{4}$ ,  $|k| \leq s_0$ , satisfying

$$\mathcal{A}_q(k) = \mathcal{A}_0 + \tilde{\mathcal{A}}(k).$$

Let us consider the following decomposition of the operator  $\mathcal{B}_q$ ,

$$(3.12) \quad \mathcal{B}_q = \mathcal{B}_q \mathbf{1}_{[0, \frac{1}{2}s]}(\mathcal{B}_q) + \mathcal{B}_q \mathbf{1}_{[\frac{1}{2}s, \infty]}(\mathcal{B}_q).$$

Since we have  $\left\| (iJ/k) \mathcal{B}_q \mathbf{1}_{[0, \frac{1}{2}s]}(\mathcal{B}_q) + \tilde{\mathcal{A}}(k) \right\| < \frac{3}{4}$  for  $0 < s < |k| < s_0$ , then

$$(3.13) \quad \left( I + \mathcal{T}_W(\lambda_q(k)) \right) = (I + \mathcal{K}(k, s)) \left( I + \frac{J}{k} \mathcal{B}_q \mathbf{1}_{[0, \frac{1}{2}s]}(\mathcal{B}_q) + \tilde{\mathcal{A}}(k) \right),$$

where the operator  $\mathcal{K}(k, s)$  is given by

$$\mathcal{K}(k, s) := \left( \frac{J}{k} \mathcal{B}_q \mathbf{1}_{[\frac{1}{2}s, \infty]}(\mathcal{B}_q) + \mathcal{A}_0 \right) \left( I + \frac{J}{k} \mathcal{B}_q \mathbf{1}_{[0, \frac{1}{2}s]}(\mathcal{B}_q) + \tilde{\mathcal{A}}(k) \right)^{-1}.$$

Note that  $\mathcal{K}(k, s)$  is a finite-rank operator, and (3.9) implies that its rank is of order

$$\mathcal{O}\left(\text{Tr} \mathbf{1}_{(\frac{1}{2}s, \infty)}(\mathcal{B}_q) + 1\right) = \mathcal{O}\left(\text{Tr} \mathbf{1}_{(s, \infty)}(p_q |W| p_q) + 1\right).$$

Moreover, its norm is of order  $\mathcal{O}(|k|^{-1})$ . On the other hand, since  $\|(J/k) \mathcal{B}_q \mathbf{1}_{[0, \frac{1}{2}s]}(\mathcal{B}_q) + \tilde{\mathcal{A}}(k)\| < 1$  for  $0 < s < |k| < s_0$ , then by [14, Theorem 4.4.3] we have

$$\text{Ind}_\gamma \left( I + (J/k) \mathcal{B}_q \mathbf{1}_{[0, \frac{1}{2}s]}(\mathcal{B}_q) + \tilde{\mathcal{A}}(k) \right) = 0.$$

Now (3.11) follows immediately applying to (3.13) the properties on the index of a finite meromorphic function recalled in the Appendix. Consequently, Proposition 3.2 combined with (3.11) show that  $\lambda_q(k)$  is a discrete eigenvalue of  $\mathcal{H}$  if and only if  $k$  is a zero of the determinant  $\mathcal{D}(k, s)$  defined by (3.10).

(iii) Identity (3.13) implies that for  $0 < s < |k| < s_0$ , we have

$$(3.14) \quad I + \mathcal{K}(k, s) = \left( I + \mathcal{T}_W(\lambda_q(k)) \right) \left( I + \frac{J}{k} \mathcal{B}_q \mathbf{1}_{[0, \frac{1}{2}s]}(\mathcal{B}_q) + \tilde{\mathcal{A}}(k) \right)^{-1}.$$

Let  $\mathcal{R}_0(\lambda)$  and  $\mathcal{R}(\lambda)$  be respectively the resolvents of the operators  $\mathcal{H}_0$  and  $\mathcal{H}$ . From the resolvent equation, it can be easily checked that

$$\left( I + J|W|^{1/2} \mathcal{R}_0(\lambda) |W|^{1/2} \right) \left( I - J|W|^{1/2} \mathcal{R}(\lambda) |W|^{1/2} \right) = I.$$

So, if  $\lambda_q(k)$  is not a discrete eigenvalue of  $\mathcal{H}$ , then we have

$$\left( I + \mathcal{T}_W(\lambda_q(k)) \right)^{-1} = I - J|W|^{1/2} (\mathcal{H} - \lambda_q(k))^{-1} |W|^{1/2}.$$

This together with (3.14) implies that the operator  $I + \mathcal{K}(k, s)$  is invertible for  $0 < s < |k| < s_0$ . Moreover, if  $|\Im(k)| > \varsigma > 0$ , then

$$\begin{aligned} \left\| (I + \mathcal{K}(k, s))^{-1} \right\| &= \mathcal{O} \left( 1 + \left\| |W|^{1/2} (\mathcal{H} - \lambda_q(k))^{-1} |W|^{1/2} \right\| \right) \\ &= \mathcal{O} \left( 1 + \text{dist}(\lambda_q(k), \overline{N(\mathcal{H})})^{-1} \right) \\ &= \mathcal{O}(\varsigma^{-1}), \end{aligned}$$

with the help of [6, Lemma 9.3.14]. This concludes the proof of the proposition.  $\blacksquare$

Proposition 3.4 above implies that for  $0 < s < |k| < s_0$ , we have

$$\begin{aligned} (3.15) \quad \mathcal{D}(k, s) &= \frac{\mathcal{O}(\text{Tr} \mathbf{1}_{(s, \infty)}(p_q |W| p_q) + 1)}{\prod_{j=1}^{\infty} (1 + \lambda_j(k, s))} \\ &= \mathcal{O}(1) \exp \left( \mathcal{O}(\text{Tr} \mathbf{1}_{(s, \infty)}(p_q |W| p_q) + 1) |\ln s| \right), \end{aligned}$$

where  $\lambda_j(k, s)$  are the eigenvalues of the operator  $\mathcal{K} := K(k, s)$  satisfying  $|\lambda_j(k, s)| = \mathcal{O}(s^{-1})$ . Let  $k$  satisfying  $\Lambda_q - k$  is not a discrete eigenvalue of  $\mathcal{H}$  with  $|\Im(k)| > \varsigma > 0$  and  $0 < s < |k| < s_0$ . Then,

$$\mathcal{D}(k, s)^{-1} = \det(I + K)^{-1} = \det(I - K(I + K)^{-1}).$$

So as in (3.15), it can be checked that

$$(3.16) \quad |\mathcal{D}(k, s)| \geq C \exp \left( -C(\text{Tr} \mathbf{1}_{(s, \infty)}(p_q |W| p_q) + 1)(|\ln \varsigma| + |\ln s|) \right).$$

Consider the discrete eigenvalues  $\lambda_q(k)$  for  $k \in \{r < |k| < 2r\} \subset D(0, \varepsilon)^*$  with  $r > 0$ . Due to their discontinuous distribution, there exists a simply connected domain  $\Delta \subset \{r < |k| < 2r\}$  containing all the eigenvalues  $\lambda_q(k)$ , and satisfying  $\Lambda_q - k$  is not a discrete eigenvalue for any  $k \in \partial\Omega$ . Now, let us apply Lemma 4.1 with the function  $g(k) := \mathcal{D}(k, r)$  and with some  $k_0 \in \Delta$  such that  $\text{dist}(\lambda_q(k_0), \overline{N(\mathcal{H})}) > \varsigma > 0$ . By Proposition 3.4, look at the zeros of the function  $g(k)$  on a sub-domain  $\Delta' \Subset \Delta$  is equivalent to look at the discrete eigenvalues  $\lambda_q(k)$  of the operator  $\mathcal{H}$ . This together with (3.15) and (3.16) give estimate (1.12) of Theorem 1.2.



**3.3. Proof of Theorem 1.3.** First, let us reformulate Proposition 3.2 with the help of characteristic value terminology, see Definition 4.1.

**Proposition 3.5.** *Let  $\mathcal{T}_W(\lambda_q(k))$  be the operator defined in Proposition 3.1. Then for  $k_0 \in D(0, \epsilon)^*$  with  $\epsilon$  small enough, the following assertions are equivalent:*

- (i)  $\lambda_q(k_0) := \Lambda_q - k_0$  is a discrete eigenvalue of  $\mathcal{H}$ ,
- (ii)  $k_0$  is a characteristic value of  $I + \mathcal{T}_W(\lambda_q(k))$ .

Moreover, we have

$$(3.17) \quad \text{mult}(\lambda_q(k_0)) = \text{mult}(k_0)$$

where the right side hand of (3.17) is the multiplicity of the characteristic value  $k_0$  defined by (4.6).

Hence, if  $W = e^{i\alpha}V$  as in (1.13), the study of the discrete eigenvalues  $\lambda_q(k) := \Lambda_q - k$  near  $\Lambda_q$ ,  $q \in \mathbb{N}$  can be reduced to the study of characteristic values of

$$I + \mathcal{T}_{e^{i\alpha}V}(\lambda_q(k)) = I - e^{i\alpha} \frac{\mathcal{A}_q(k)}{k},$$

where the operator  $\mathcal{A}_q(k)$  is defined by (1.14). In particular, for  $J = \pm$ , we have  $\mp \mathcal{A}_q(0) = |V|^{\frac{1}{2}} p_q |V|^{\frac{1}{2}}$  which is non-negative.

So, point (i) of Theorem 1.3 is an immediate consequence of Lemma 4.2 with  $z = k/e^{i\alpha}$ . More precisely, the discrete eigenvalues  $\lambda_q(k)$  satisfy

$$(3.18) \quad \mp \Re \left( \frac{k}{e^{i\alpha}} \right) \geq 0, \quad k \in \mp e^{i\alpha} \mathcal{C}_\delta(r, r_0),$$

for any  $\delta > 0$ , with the sector  $\mathcal{C}_\delta(r, r_0)$  defined by (1.17).

Let us prove (ii). From (i), we have that the discrete eigenvalues  $\lambda_q(k)$  are concentrated in the sector  $\{k \in D(0, \epsilon)^* : \mp k/e^{i\alpha} \in \mathcal{C}_\delta(r, r_0)\}$  for any  $\delta > 0$ . In particular, for  $r \searrow 0$ , we have

$$(3.19) \quad \sum_{\substack{\lambda_q(k) \in \sigma_{\text{disc}}(\mathcal{H}) \\ r < |k| < r_0}} \text{mult}(\lambda_q(k)) = \sum_{\substack{\lambda_q(k) \in \sigma_{\text{disc}}(\mathcal{H}) \\ \mp k/e^{i\alpha} \in \mathcal{C}_\delta(r, r_0)}} \text{mult}(\lambda_q(k)) + \mathcal{O}(1) \\ = \mathcal{N}(\mathcal{C}_\delta(r, r_0)) + \mathcal{O}(1),$$

where  $\mathcal{N}(\cdot)$  is the quantity defined by (4.7). Let  $n(\cdot)$  be the quantity defined by (4.8) with  $T(0) = \mp \mathcal{A}_q(0)$ , so that

$$(3.20) \quad n([r, r_0]) = \text{Tr} \mathbf{1}_{(r, \infty)} \left( |V|^{\frac{1}{2}} p_q |V|^{\frac{1}{2}} \right) + \mathcal{O}(1) \\ = \text{Tr} \mathbf{1}_{(r, \infty)} (p_q |V| p_q) + \mathcal{O}(1)$$

Thus, point (ii) is a direct consequence of Lemma 4.3 together with (3.19) and (3.20).

For the point (iii), the assumption  $\text{Tr} \mathbf{1}_{(r, \infty)} (p_q |V| p_q) = \phi(r)(1 + o(1))$  as  $r \searrow 0$  implies that  $n([r, 1]) = \phi(r)(1 + o(1))$  as  $r \searrow 0$ . Then it follows from Lemma 4.4 together with (3.19). This concludes the proof the theorem.

## 4. APPENDIX

For the transparency of the presentation, we recall in this appendix some auxiliary results.

**4.1. Notion of index of a finite meromorphic operator.** In this subsection, we recall the notion of index (with respect to a positively oriented contour) of a holomorphic function and a finite meromorphic operator-valued function, see e.g. [3, Definition 2.1].

Let  $f$  be a holomorphic function in a neighborhood of a contour  $\gamma$ . The index of  $f$  with respect to the contour  $\gamma$  is defined by

$$(4.1) \quad \text{ind}_\gamma f := \frac{1}{2i\pi} \int_\gamma \frac{f'(z)}{f(z)} dz.$$

Note that if  $f$  is holomorphic in a domain  $\Omega$  such that  $\partial\Omega = \gamma$ , then by residues theorem  $\text{ind}_\gamma f$  coincides with the number of zeros of the function  $f$  in  $\Omega$ , counted according to their multiplicity. Now let  $D \subseteq \mathbb{C}$  be a connected open set,  $Z \subset D$  a pure point and closed subset and  $A : \overline{D} \setminus Z \rightarrow \text{GL}(E)$  a finite meromorphic operator-valued function and Fredholm at each point of  $Z$ . The index of  $A$  with respect to the contour  $\partial\Omega$  is defined by

$$(4.2) \quad \text{Ind}_{\partial\Omega} A := \frac{1}{2i\pi} \text{tr} \int_{\partial\Omega} A'(z) A(z)^{-1} dz = \frac{1}{2i\pi} \text{tr} \int_{\partial\Omega} A(z)^{-1} A'(z) dz.$$

We have the following properties:

$$(4.3) \quad \text{Ind}_{\partial\Omega} A_1 A_2 = \text{Ind}_{\partial\Omega} A_1 + \text{Ind}_{\partial\Omega} A_2,$$

and if  $K(z)$  is in the trace class operators, then

$$(4.4) \quad \text{Ind}_{\partial\Omega} (I + K) = \text{ind}_{\partial\Omega} \det(I + K).$$

For more details, see [14, Chap. 4].

**4.2. Characteristic values of holomorphic operators.** In this subsection, we recall some results due to J.-F. Bony, V. Bruneau and G. Raikov [3] on characteristic values of holomorphic operators.

The following lemma contains a version of the well-known Jensen inequality.

**Lemma 4.1.** [2, Lemma 6] *Let  $\Delta$  be a simply connected sub-domain of  $\mathbb{C}$  and let  $g$  be a holomorphic function in  $\Delta$  with continuous extension to  $\overline{\Delta}$ . Assume there exists  $\lambda_0 \in \Delta$  such that  $g(\lambda_0) \neq 0$  and  $g(\lambda) \neq 0$  for  $\lambda \in \partial\Delta$ , the boundary of  $\Delta$ . Let  $\lambda_1, \lambda_2, \dots, \lambda_N \in \Delta$  be the zeros of  $g$  repeated according to their multiplicity. For any domain  $\Delta' \Subset \Delta$ , there exists  $C' > 0$  such that  $N(\Delta', g)$ , the number of zeros  $\lambda_j$  of  $g$  contained in  $\Delta'$ , satisfies*

$$(4.5) \quad N(\Delta', g) \leq C' \left( \int_{\partial\Delta} \ln|g(\lambda)| d\lambda - \ln|g(\lambda_0)| \right).$$

Let  $\mathcal{D}$  be a domain of  $\mathbb{C}$  containing 0 and let us consider an holomorphic operator-valued function  $T : \mathcal{D} \rightarrow S_\infty$ .

**Definition 4.1.** For a domain  $\Omega \subset \mathcal{D} \setminus \{0\}$ , a complex number  $z \in \Omega$  is an characteristic value of  $z \mapsto \mathcal{T}(z) := I - \frac{T(z)}{z}$  if the operator  $\mathcal{T}(z)$  is not invertible. The multiplicity of a characteristic value  $z_0$  is defined by

$$(4.6) \quad \text{mult}(z_0) := \text{Ind}_\gamma(I - \mathcal{T}(\cdot)),$$

where  $\gamma$  is a small contour positively oriented containing  $z_0$  as the unique point  $z$  satisfying  $\mathcal{T}(z)$  is not invertible.

Denote by

$$\mathcal{Z}(\Omega) := \{z \in \Omega : \mathcal{T}(z) \text{ is not invertible}\}.$$

If there exists  $z_0 \in \Omega$  such that  $\mathcal{T}(z_0)$  is not invertible, then  $\mathcal{Z}(\Omega)$  is pure point, see e.g. [13, Proposition 4.1.4]. So we set

$$(4.7) \quad \mathcal{N}(\Omega) := \#\mathcal{Z}(\Omega).$$

Assume that  $T(0)$  is self-adjoint. Introduce  $\Omega \Subset \mathbb{C} \setminus \{0\}$  and let  $\mathcal{C}_\delta(r, r_0)$  be the domain defined by (1.17). Put

$$(4.8) \quad n(\Lambda) := \text{Tr } \mathbf{1}_\Lambda(T(0)),$$

the number of eigenvalues of the operator  $T(0)$  lying in the interval  $\Lambda \subset \mathbb{R}^*$ , and counted with their multiplicity. Denoted by  $\Pi_0$  the orthogonal onto  $\ker T(0)$ .

**Lemma 4.2.** [3, Corollary 3.4] *Let  $T$  be as above with  $I - T'(0)\Pi_0$  invertible. Let  $\Omega \Subset \mathbb{C} \setminus \{0\}$  be a bounded domain such that  $\partial\Omega$  is smooth and transverse to the real axis at each point of  $\partial\Omega \cap \mathbb{R}$ .*

(i) *If we have  $\Omega \cap \mathbb{R} = \emptyset$ , then for  $s$  sufficiently small,  $\mathcal{N}(s\Omega) = 0$ . So, the characteristic values  $z \in \mathcal{Z}(\mathcal{D})$  satisfy  $|\Im(z)| = o(|z|)$  near 0.*

(ii) *Moreover, if  $T(0)$  satisfy  $\pm T(0) \geq 0$ , then the characteristic values  $z$  satisfy respectively  $\pm \Re(z) \geq 0$  near 0.*

(iii) *If the operator  $T(0)$  is of finite rank, then in a pointed neighborhood of 0, there are no characteristic values. If  $T(0)\mathbf{1}_{[0, +\infty)}(\pm T(0))$  is of finite rank, then in a neighborhood of 0 intersected with  $\{\pm \Re(z) > 0\}$ , there are no characteristic values respectively.*

**Lemma 4.3.** [3, Corollary 3.9] *Suppose that the assumptions of Lemma 4.2 happens. Assume that there exists a constant  $\gamma > 0$  such that*

$$n([r, 1]) = \mathcal{O}(r^{-\gamma}), \quad r \searrow 0,$$

*with  $n([r, 1])$  growing unboundedly as  $r \searrow 0$ . Then there exists a positive sequence  $(r_k)_k$  which tends to 0 such that*

$$(4.9) \quad \mathcal{N}(\mathcal{C}_\delta(r_k, 1)) = n([r_k, 1])(1 + o(1))$$

*as  $k \rightarrow \infty$*

**Lemma 4.4.** [3, Corollary 3.11] *Suppose that the assumptions of Lemma 4.2 happens. If we have*

$$n([r, 1]) = \Phi(r)(1 + o(1))$$

*as  $r \searrow 0$ , with  $\phi(r(1 \pm \nu)) = \phi(r)(1 + o(1) + \mathcal{O}(\nu))$  for any  $\nu > 0$  small enough, then*

$$(4.10) \quad \mathcal{N}(\mathcal{C}_\alpha(r, 1)) = \Phi(r)(1 + o(1))$$

as  $r \searrow 0$ .

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FACULTAD DE MATEMÁTICAS PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE, AV.  
VICUNA MACKENNA 4860, SANTIAGO DE CHILE

*E-mail address:* `disambou@mat.uc.cl`